"Inequalities Concerning Polar Derivative of Polynomials"

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Abstract: Let $p(z)$ be a polynomial of degree n and let α be any real or complex number, then the polar derivative of $p(z)$ denoted by $D_\alpha\, p(z)$, is defined as

$$D_\alpha\, p(z) = n\, p(z) + (\alpha - z)\, p'(z)$$

The polynomial $D_\alpha\, p(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $p'(z)$ of $p(z)$ in the sense that

$$\underset{\alpha\to\infty}{Lim}\ \frac{D_\alpha\, p(z)}{\alpha} = p'(z)$$

In this paper we prove interesting results for the polar derivative of a polynomial which not only improve upon some earlier known results in the same area but also improve upon a result on ordinary derivative for polynomials in particular case.

Key-Words: Polynomials; Polar derivative; Inequalities; Zeros.

AMS Subject Classification: 30A10, 30C10

1. Introduction and statement of results:

Let $p(z)$ be a polynomial of degree n, then according to a famous result known as Bernstein's inequality (for reference see [2]), we have

Theorem A. *If $p(z)$ is a polynomial of degree n, then*

$$\max_{|z|=1}|p'(z)| \le n \max_{|z|=1}|p(z)| . \tag{1.1}$$

The result is best possible and equality holds for $p(z)=\lambda z^n$, $\lambda(\neq 0)$ being a complex number.

For the class of polynomials having no zeros in $|z|<1$, the following result was conjectured by Erd $\ddot{o}$ s and later verified by Lax [5].

Theorem B. *If $p(z)$ is a polynomial of degree n having no zeros in $|z|<1$, then*

$$\max_{|z|=1}|p'(z)| \le \frac{n}{2} \max_{|z|=1}|p(z)| . \tag{1.2}$$

The result is best possible and equality in (1.2) *occurs for $p(z)=\lambda+\mu z^n$, where $|\lambda|=|\mu|$.*

As a generalization of Theorem B, Malik [6] proved the following

Theorem C. *If $p(z)$ is a polynomial of degree n having no zeros in $|z|<k$, $k\ge 1$, then*

$$\max_{|z|=1} |p'(z)| \le \frac{n}{1+k} \max_{|z|=1} |p(z)| . \tag{1.3}$$

The result is sharp and extremal polynomial is $p(z)=(z+k)^n$.

For the class of polynomials not vanishing in the disk $|z|<k$, $k\ge 1$, Aziz [1] extended Theorem C to the polar derivative and proved the following result.

Theorem D. *If* $p(z)$ *is a polynomial of degree* n*, having no zeros in the disk* $|z|<k,\ k\geq 1$*, then for every real or complex number* β *with* $|\beta|\geq 1$*,*

$$\max_{|z|=1}\left|D_{\beta}p(z)\right| \leq \frac{n(k+|\beta|)}{1+k}\max_{|z|=1}\left|p(z)\right|. \tag{1.4}$$

The result is sharp and equality in (1.4) *holds for the polynomial* $p(z)=(z+k)^n$ *with real* $\beta\geq 1$.

Recently, Govil and Mc Tume [3], for the class of polynomials having all the zeros in the region $|z|\leq k,\ k\leq 1$, proved the following

Theorem E. *If* $p(z)=\sum_{\nu=0}^{n}a_{\nu}z^{\nu}$ *is a polynomial of degree* n*, having all its zeros in* $|z|\leq k,\ k\leq 1$*, then for every real or complex number* α *with* $|\alpha|\geq L$*,* $L=\dfrac{nk^2|a_n|+|a_{n-1}|}{n|a_n|+|a_{n-1}|}$

,

$$\max_{|z|=1}\left|D_{\alpha}p(z)\right| \geq \frac{n(|\alpha|-L)}{1+k}\max_{|z|=1}\left|p(z)\right|. \tag{1.5}$$

In this paper we prove the following result which improves upon Theorem E.

Theorem 1. *If* $p(z)=\sum_{\nu=0}^{n}a_{\nu}z^{\nu}$ *is a polynomial of degree* n *having all its zeros in* $|z|<k,\ k\leq 1$*, then for every real or complex number* α *with* $|\alpha|\geq 1+k+L$*, where* $L=\dfrac{nk^2|a_n|+|a_{n-1}|}{n|a_n|+|a_{n-1}|}$*, then for* $|z|=1$*,*

$$\max_{|z|=1}|D_\alpha p(z)| \geq \frac{n(|\alpha|-L)}{(1+k)}\max_{|z|=1}|p(z)| + n\left(\frac{|\alpha|-(1+k+L)}{(1+k)}\right)\min_{|z|=k}|p(z)|. \tag{1.6}$$

Next we prove a result that, in particular case, gives an improvement of Theorem 1. Here we assumed that some zeros of $p(z)$ are located at the origin. More precisely, we prove

Theorem 2. *If* $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having all its zeros in* $|z|\leq k,\ k\leq 1$*, with s-fold zeros at origin, then for every real or complex number* α *with* $|\alpha|\geq L$*, we have*

$$\max_{|z|=1}|D_\alpha p(z)| \geq \frac{(|\alpha|-L)(n+sk)}{(1+k)}\max_{|z|=1}|p(z)| + \left\{n+\frac{(n-s)(|\alpha|-L)}{k^s(1+k)}\right\}\min_{|z|=k}|p(z)| \tag{1.7}$$

Where L *is same as defined in Theorem 1.*

Remark 3. As earlier said for $s=0$, Theorem 2 gives an improvement of Theorem 1. To see this, firstly we will show $k\geq L$. If $z_i,\ 1\leq i\leq n$, are the zeros of $p(z)$, then $|z_i|\leq k,\ 1\leq i\leq n$ and

$$\left|\frac{a_{n-1}}{a_n}\right| = |z_1+z_2+\cdots+z_n| \leq nk$$

$$|a_{n-1}| \leq n|a_n|k \tag{1.8}$$

Since $k\leq 1$, inequality (1.8) gives

$$(1-k)|a_{n-1}| \leq (1-k)n|a_n|k\,.$$

Which is equivalent to $L = \frac{n k^2 |a_n| + |a_{n-1}|}{n|a_n| + |a_{n-1}|} \le k$

or $$k \ge L. \tag{1.9}$$

In view of $k \ge L$, we see that Theorem 2 (for s=0) gives an improvement of Theorem 1. To see this

$$\frac{n(|\alpha| - L)}{1+k} \|p\| + n\left(\frac{1+k+|\alpha| - L}{1+k} \right) m \ge \frac{n(|\alpha| - L)}{(1+k)} \|p\| + n\left\{ \frac{|\alpha| - (1+k+L)}{1+k} \right\} m$$

This is equivalent to

$$0 \ge -2(1+k)m$$

or $$2(1+k)m \ge 0$$

which is always true. Hence the Remark 3.4 is true. Also we note that Theorem 3.3 is valid for $|\alpha| \ge 1+k+L$. While Theorem 3.4 is valid for $|\alpha| \ge L$, which is a bigger region.

2. Lemmas:

For the proof of the theorems, we need the following lemmas.

Lemma 2.1. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n, having all its zeros in $|z| \le k$, $k \le 1$, then on $|z| = 1$*

$$L\,|p'(z)| \ge |q'(z)|, \tag{2.1}$$

where

$$L = \frac{n|a_n|k^2 + |a_{n-1}|}{n|a_n| + |a_{n-1}|} \, .$$

The above result is due to Govil and Mc Tume [3].

We improve upon Lemma 2.1 as follows

Lemma 2.2. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree n, having all its zeros in* $|z| \le k,\ k \le 1$*, then on* $|z| = 1$ *, we have*

$$L\left|p'(z)\right| \ge \left|q'(z)\right| + mn, \tag{2.2}$$

where

$$m = \min_{|z|=k} |p(z)|$$

and L *is same as defined in Lemma* 2.1.

Proof of Lemma 2.2. Since $p(z)$ has all its zeros in $|z| \le k,\ k \le 1$, the polynomial $F(z) = p(z) + \lambda m,\ |\lambda| < 1,\quad m = \min_{|z|=k} |p(z)|$, will also have all its zeros in the same domain i.e. $|z| \le k,\ k \le 1$. Let

$$Q(z) = z^n \overline{F\left(\frac{1}{\bar{z}}\right)}$$

$$= z^n \overline{p\left(\frac{1}{\bar{z}}\right) + \lambda m}$$

$$= z^n \overline{p\left(\frac{1}{\bar{z}}\right)} + z^n \bar{\lambda} m$$

$$Q(z) = q(z) + z^n \bar{\lambda} m.$$

Applying Lemma2.1 for $F(z)$, we have

$$L|F'(z)| \geq |Q'(z)| \quad \text{for} \quad |z| = 1$$

or

$$L|p'(z)| \geq \left|q'(z) + n z^{n-1} \bar{\lambda} m\right| \tag{2.3}$$

Now suitably choosing the argument of λ in such a way that R. H. S. of inequality (2.3) on $|z| = 1$, becomes

$$\left|q'(z) + n z^{n-1} \bar{\lambda} m\right| = |q'(z)| + n|\lambda| m$$

Hence inequality (2.3) becomes

$$L|p'(z)| \geq |q'(z)| + n|\lambda| m .$$

Finally, letting $|\lambda| \to 1$, the proof of Lemma 2.2 is completed.

Lemma 2.3. *If all the zeros of a polynomial* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *of degree n, lie in* $|z| \leq k$, $k \leq 1$, *with s-fold zeros at origin, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{(n+sk)}{1+k} \max_{|z|=1} |p(z)| + \frac{(n-s)}{k^s(1+k)} \min_{|z|=k} |p(z)|. \tag{2.4}$$

The above result is due to Jain [4, Corollary 1.7].

3. Proof of the Theorems:

Proof of Theorem 1: Since $p(z)$ has all of its zeros in $|z| < k,\ k \le 1$, by Rouche's Theorem, the polynomial $p(z) + \lambda m$, where $|\lambda| < 1$ and $m = \min_{|z|=k} |p(z)|$ also has all its zeros in same domain i.e. $|z| < k,\ k \le 1$. Therefore applying inequality (1.5) to $p(z) + \lambda m$, we have,

$$\max_{|z|=1} |D_\alpha \{p(z) + \lambda m\}| \quad \ge \frac{n(|\alpha| - L)}{1+k} \max_{|z|=1} |p(z) + \lambda m \quad |$$

or

$$\max_{|z|=1} |D_\alpha\, p(z) + \lambda mn| \quad \ge \frac{n(|\alpha| - L)}{1+k} \max_{|z|=1} |p(z) + \lambda m \quad |$$

$$\max_{|z|=1} |D_\alpha\, p(z)| + |\lambda|\, mn \quad \ge \frac{n(|\alpha| - L)}{1+k} \max_{|z|=1} |p(z) + \lambda m \quad | \tag{2.1}$$

Now choosing the argument of λ suitably such that R.H.S. of inequality (2.1) becomes

$$\max_{|z|=1} |p(z) + \lambda m| \quad = \max_{|z|=1} |p(z)| + |\lambda| m \tag{2.2}$$

Combining equality (2.1) and inequality (2.2), we get

$$\max_{|z|1} |D_\alpha\, p(z)| + |\lambda|\, mn \quad \ge \frac{n(|\alpha| - L)}{1+k} \left\{\max_{|z|=1} |p(z)| + |\lambda| m\right\}$$

or

$$\max_{|z|1}|D_\alpha p(z)| \geq \frac{n(|\alpha|-L)}{1+k}\left\{\max_{|z|=1}|p(z)|\right\}+|\lambda|mn\left\{\frac{|\alpha|-L}{1+k}-1\right\}$$

$$=\frac{n(|\alpha|-L)}{1+k}\max_{|z|=1}|p(z)| +|\lambda|n\left(\frac{|\alpha|-(1+k+L)}{1+k}\right)\min_{|z|=k}|p(z)|.$$

Finally letting $|\lambda|\to 1$, we get the desired result.

Proof of Theorem 2: By definition

$$D_\alpha p(z)=np(z)+(\alpha-z)p'(z)$$

which implies

$$|D_\alpha p(z)|=|\alpha p'(z)+np(z)-zp'(z)|$$

$$\geq|\alpha||p'(z)|-|np(z)-zp'(z)| \quad . \tag{3.3}$$

Let $q(z)\equiv z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$, then on $|z|=1$,

$$|np(z)-zp'(z)|=|q'(z)| \quad . \tag{3.4}$$

Combining inequalities (3.3) and (3.4), we get

$$|D_\alpha p(z)|\geq|\alpha||p'(z)|-|q'(z)| \quad \text{on} \quad |z|=1 . \tag{3.5}$$

Now using Lemma 2.2 in (3.5), we get

$$\begin{aligned}|D_\alpha p(z)| &\geq |\alpha||p'(z)| - \left\{L|q'(z)| - mn\right\} \\ &= (|\alpha| - L)|p'(z)| + mn \end{aligned} \tag{3.6}$$

where *m is defined as in Lemma 2.2.*

Now for polynomial having all its zeros in $|z| < k,\ k \leq 1$, with s-fold zeros at origin, applying Lemma 2.3 in (3.6), we get

$$|D_\alpha p(z)| \geq (|\alpha| - L)\left\{\frac{(n+sk)}{1+k}\max_{|z|=1}|p(z)| + \frac{(n-s)}{k^s(1+k)}m\right\} + mn.$$

Equivalently

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| &\geq \frac{(|\alpha| - L)(n+sk)}{(1+k)}\max_{|z|=1}|p(z)| \\ &+ \left\{n + \frac{(n-s)(|\alpha| - L)}{k^s(1+k)}\right\}\min_{|z|=k}|p(z)|.\end{aligned}$$

Thus the theorem is proved completely.

Acknowledgement: The author is highly thankful to anonymous referee for his

valuable suggestions.

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